# Theoretical foundations – 2.4 Random utility theory

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*Mathematical derivation of the choice model*

We derive here the general random utility model. Although the derivation is quite straightforward, it is also technical. It may be skipped without loss of continuity in the course.

Consider the choice model with $J_n$ alternatives

$$P(i|\mathcal{C}_n) = \Pr(U_{in} \geq U_{jn}, \forall j = 1, \ldots, J_n), \quad (1)$$

where

$$U_{in} = V_{in} + \varepsilon_{in}. \quad (2)$$

Denote by

$$\varepsilon_n = (\varepsilon_{1n}, \ldots, \varepsilon_{J_n n})$$

the vector of $J_n$ error terms. If $\varepsilon_n$ is a multivariate random variable with CDF $F_{\varepsilon_n}(\varepsilon_1, \ldots, \varepsilon_{J_n})$ and pdf

$$f_{\varepsilon_n}(\varepsilon_1, \ldots, \varepsilon_{J_n}) = \frac{\partial^{J_n} F}{\partial \varepsilon_1 \cdots \partial \varepsilon_{J_n}}(\varepsilon_1, \ldots, \varepsilon_{J_n}), \quad (3)$$

then

$$P_n(i|\mathcal{C}_n) = \int_{\varepsilon_i=-\infty}^{+\infty} \int_{\varepsilon_1=-\infty}^{V_{in}-V_{1n}+\varepsilon_i} \cdots \int_{\varepsilon_{i-1}=-\infty}^{V_{in}-V_{i-1n}+\varepsilon_i} \int_{\varepsilon_{i+1}=-\infty}^{V_{in}-V_{i+1n}+\varepsilon_i} \cdots \int_{\varepsilon_{J_n}=-\infty}^{V_{1n}-V_{J_n n}+\varepsilon_1} f_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{J_n}) d\varepsilon. \quad (4)$$

and

$$P_n(i|\mathcal{C}_n) = \int_{\varepsilon=-\infty}^{+\infty} \frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}}{\partial \varepsilon_i}(\ldots, V_{in}-V_{(i-1)n}+\varepsilon, \varepsilon, V_{in}-V_{(i+1)n}+\varepsilon, \ldots) d\varepsilon. \tag{5}$$

Therefore, if the CDF is available in closed form, the choice model is obtained by solving a uni-dimensional integral, which can be done analytically for simple models, and numerically for more complex ones.

**Proof.** We prove the result for alternative 1 without loss of generality, in order to simplify the notations.

Using (2) into (1), we obtain

$$P(1|\mathcal{C}_n) = \Pr(V_{2n} + \varepsilon_{2n} \leq V_{1n} + \varepsilon_{1n}, \ldots, V_{J_n n} + \varepsilon_{J_n n} \leq V_{1n} + \varepsilon_{1n}), \tag{6}$$

or, gathering the random terms on one side, and the deterministic ones on the other side,

$$P_n(1|\mathcal{C}_n) = \Pr(\varepsilon_{2n} - \varepsilon_{1n} \leq V_{1n} - V_{2n}, \ldots, \varepsilon_{J_n n} - \varepsilon_{1n} \leq V_{1n} - V_{J_n n}). \tag{7}$$

We consider the following change of variables:

$$\xi_{1n} = \varepsilon_{1n},\ \xi_{jn} = \varepsilon_{jn} - \varepsilon_{1n},\ j = 2, \ldots, J_n, \tag{8}$$

that is, in matrix notations,

$$\xi_n = \begin{pmatrix} \xi_{1n} \\ \xi_{2n} \\ \vdots \\ \xi_{(J_n-1)n} \\ \xi_{J_n n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ & & \vdots & & \\ -1 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \vdots \\ \varepsilon_{(J_n-1)n} \\ \varepsilon_{J_n n} \end{pmatrix} = M\varepsilon_n.$$

Note that the determinant of the change of variables matrix $M$ is 1, so that $\varepsilon_n$ and $\xi_n$ have the same pdf. The model in the new variables becomes

$$P_n(1|\mathcal{C}_n) = \Pr(\xi_{2n} \leq V_{1n} - V_{2n}, \ldots, \xi_{J_n n} \leq V_{1n} - V_{J_n n}).$$

Therefore,

$$P_n(1|\mathcal{C}_n) = F_{\xi_{1n},\xi_{2n},\ldots,\xi_{J_n}}(+\infty, V_{1n} - V_{2n}, \ldots, V_{1n} - V_{J_n n})$$

from the definition of a cumulative distribution function. As the CDF is obtained by integrating the pdf, we have

$$P_n(1|\mathcal{C}_n) = \int_{\xi_1=-\infty}^{+\infty} \int_{\xi_2=-\infty}^{V_{1n}-V_{2n}} \cdots \int_{\xi_{J_n}=-\infty}^{V_{1n}-V_{J_n n}} f_{\xi_{1n},\xi_{2n},\ldots,\xi_{J_n}}(\xi_1, \xi_2, \ldots, \xi_{J_n}) d\xi.$$

Now we come back to the original variables, exploiting the fact that the pdf of $\xi_n$ and $\varepsilon_n$ are identical:

$$P_n(1|\mathcal{C}_n) = \int_{\varepsilon_1=-\infty}^{+\infty} \int_{\varepsilon_2=-\infty}^{V_{1n}-V_{2n}+\varepsilon_1} \cdots \int_{\varepsilon_{J_n}=-\infty}^{V_{1n}-V_{J_n n}+\varepsilon_1} f_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{J_n}) d\varepsilon.$$

By integrating over all dimensions except the first one, we obtain:

$$P_n(1|\mathcal{C}_n) = \int_{\varepsilon_1=-\infty}^{+\infty} \frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}}{\partial \varepsilon_1}(\varepsilon_1, V_{1n} - V_{2n} + \varepsilon_1, \ldots, V_{1n} - V_{J_n n} + \varepsilon_1) d\varepsilon_1.$$

□

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*Practice quiz.*

Consider the general choice model

$$P_n(i|\mathcal{C}_n) = \int_{\varepsilon=-\infty}^{+\infty} \frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}}{\partial \varepsilon_i}(\ldots, V_{in}-V_{(i-1)n}+\varepsilon, \varepsilon, V_{in}-V_{(i+1)n}+\varepsilon, \ldots)d\varepsilon.$$

Derive it for alternative $i$ in the binary case with $\mathcal{C}_n = \{i, j\}$ and the CDF of the error terms is given by

$$F_\varepsilon(\varepsilon_i, \varepsilon_j) = e^{-e^{-\varepsilon_i}} e^{-e^{-\varepsilon_j}}. \tag{1}$$

## Hint

The change of variable $t = -e^{-\varepsilon}$ conveniently simplifies the integral.

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*Solution of the practice quiz.*

The CDF of the error terms is given by

$$F_\varepsilon(\varepsilon_i, \varepsilon_j) = e^{-e^{-\varepsilon_i}} e^{-e^{-\varepsilon_j}}. \tag{1}$$

We have

$$P(i|\{i,j\}) = \int_{\varepsilon=-\infty}^{+\infty} \frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon, V_i - V_j + \varepsilon) d\varepsilon. \tag{2}$$

From (1), we have

$$\frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon_i, \varepsilon_j) = e^{-e^{-\varepsilon_i}} e^{-e^{-\varepsilon_j}} e^{-\varepsilon_i}. \tag{3}$$

Therefore,

$$\frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon, V_i - V_j + \varepsilon) = e^{-e^{-\varepsilon}} e^{-e^{-(V_i - V_j + \varepsilon)}} e^{-\varepsilon} = e^{-e^{-\varepsilon}} e^{-Ke^{-\varepsilon}} e^{-\varepsilon} \tag{4}$$

where

$$K = \exp(-(V_i - V_j)). \tag{5}$$

Therefore,

$$P(i|\{i,j\}) = \int_{\varepsilon=-\infty}^{+\infty} e^{-e^{-\varepsilon}} e^{-Ke^{-\varepsilon}} e^{-\varepsilon} d\varepsilon. \tag{6}$$

Define

$$t = -e^{-\varepsilon},\ dt = e^{-\varepsilon} d\varepsilon,$$

to obtain

$$P(i|\{i,j\}) = \int_{t=-\infty}^{0} e^{(1+K)t} dt = \frac{1}{1+K}. \tag{7}$$

Using (5), we obtain the simple expression:

$$P(i|\{i,j\}) = \frac{1}{1+\exp(-(V_i - V_j))} = \frac{e^{V_i}}{e^{V_i} + e^{V_j}}. \tag{8}$$

This happens to be the binary logit model.